Assignment 3 solutions

Note Title 30/09/2008

Assignment 3: Due Tues 30.9.2008. 10 problems: Sec 2.3 #10, 22, 28, 38, 42 and Sec. 2.5 #30, 34, 36, 42, 52.

2.3 #10 a) The equation is not true as it stands because the LHS is not defined at $x=2$, being $\frac{0}{0}$ there. 4

b) The definition of limit (by design) excludes the limit point itself, and so we may exchange the limit function for a function identical for $x \neq 2$ (but continuous). We may then use the cts function to evaluate the limit. 4

2.3 #22 $\lim\limits_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{1+h}-1\right)}{h} \cdot \frac{\left(\sqrt{1+h}+1\right)}{\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0} \frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$ 4

#28 $\lim\limits_{h\to 0} \dfrac{(3+h)^{-1}-3^{-1}}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1}{3+h}-\frac{1}{3}}{h}$

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h\cdot 3\cdot(3+h)} = \lim_{h\to 0} \frac{-1}{3(3+h)} = \frac{-1}{9}$$

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2.3 #38 Start with $-1 \le \sin\frac{\pi}{x} \le 1$.

Since e^u is monotonically increasing,

$$\frac{1}{e} = e^{-1} \le e^{\sin \pi/x} \le e^{1} = e$$

(you could also use $0 \le e^{\text{anything}}$

to get $0 \le e^{\sin \pi/x} \le e^{1}$

$$\therefore \quad \sqrt{x}\, e^{-1} \le \sqrt{x}\, e^{\sin \pi/x} \le \sqrt{x}\, e$$

note $\lim_{x\to 0^+} \sqrt{x}\, e^{-1} = 0 = \lim_{x\to 0^+} \sqrt{x}\, e$

and so $\lim_{x\to 0} \sqrt{x}\, e^{\sin \pi/x}$ must be 0, being squeezed between the two limits.

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2.3 #42

$$\lim_{x \to -2} \frac{2 - |x|}{2 + x}$$

$$= \lim_{x \to -2} \begin{cases} \dfrac{2 - (x)}{2 + x} & \text{if } x > 0 \\[2ex] \dfrac{2 - (-x)}{2 + x} & \text{if } x \leq 0 \end{cases}$$

when we take limits, we only need to care about x near to the limit point. For x near -2, clearly $x < 0$ and so only the second branch applies.

Hence $\displaystyle \lim_{x \to -2} \frac{2 - (-x)}{2 + x} = \lim_{x \to -2} \frac{2 + x}{2 + x}$

$= \underline{1}$.

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2.5 #30 $y = \ln \tan^2 x$

$\tan x$ is cts except at odd multiples of $\pi/2$

$\therefore$ $\tan^2 x$ is the same;

$\ln u$ is cts for $u > 0$

$\therefore \ln \tan^2 x$ will be cts where

BOTH $\tan^2 x$ is continuous

AND $\tan^2 x > 0$, ie $\tan x \neq 0$

$\tan x = 0$ where $\sin(x) = 0$, ie

at multiples of π (including 0)

$\therefore$ $x \neq (2k+1)\frac{\pi}{2}$ (because of tan)

<u>and</u> $x \neq n\pi$ (because of ln)

This can be summarized (combined)

into $x \neq m \cdot \frac{\pi}{2}$ for integers m.

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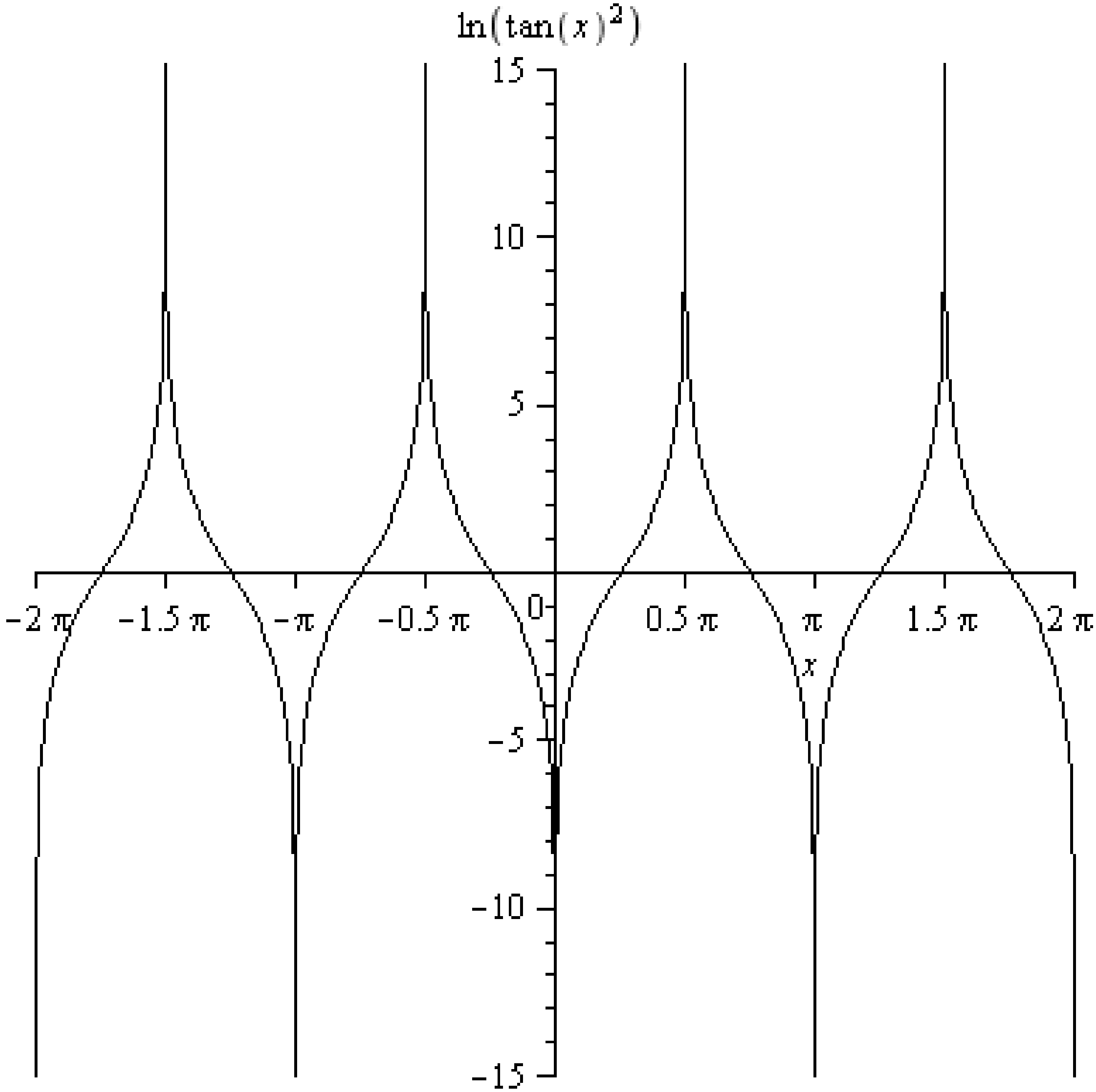

This just FYI, but the discontinuities are easily visible here at all multiples of $\pi/2$, as predicted.

2.5 #34. arctan is cts so we may take the limit inside:

$$\lim_{x \to 2} \arctan\left(\frac{x^2-4}{3x^2-6x} \right)$$

$$= \arctan\left(\lim_{x \to 2} \frac{x^2-4}{3x^2-6x} \right)$$

$$= \arctan\left(\lim_{x \to 2} \frac{(x-2)(x+2)}{3x(x-2)} \right)$$

$$= \arctan\left(\lim_{x \to 2} \frac{x+2}{3x} \right)$$

$$\boxed{= \arctan\left(\frac{2}{3} \right)}$$ since $\frac{x+2}{3x}$ is cts

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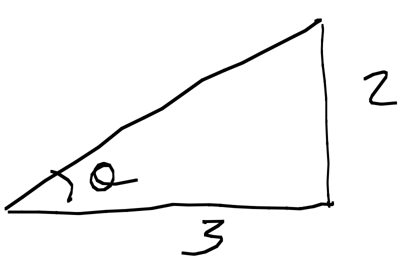
This is not a simple number:



$\theta = \arctan(2/3)$

$\doteq 0.588$ radians.

#36 $\sin(x)$ and $\cos(x)$ are cts and so we need only show that

$$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^+} f(x)$$

and both are equal to $f\left(\frac{\pi}{4}\right)$

($\pi/4$ is the one place we need to check, where the domains of each piece touch).

$$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}} \sin x = \sin \frac{\pi}{4}$$

$$= 1/\sqrt{2}$$

$$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}} \cos x = \cos \pi/4$$

$$= 1/\sqrt{2}$$

and $f(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$

(The 2^{nd} branch applies since

$$f(x) = \begin{cases} \sin(x) & x < \pi/4 \\ \cos(x) & x \geq \pi/4 \end{cases}$$)

and all three are the same.

Therefore $f(x)$ is cts. 4

2.5 #42 Typo in the problem: not defined at $x=2$. Make it

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{(x-2)(x+2)}{(x-2)} = 2+2 = 4$$

$\therefore$ we want

$$\lim_{x \to 2^+} ax^2 - bx + 3 = 4$$

ie $a \cdot 4 - b \cdot 2 + 3 = 4$

or $\boxed{4a - 2b = 1}$

and

$$\lim_{x \to 3^-} f(x) = a \cdot 3^2 - b \cdot 3 + 3$$

must be equal to

$$\lim_{x \to 3^+} f(x) = 2 \cdot 3 - a + b$$

so

$$9a - 3b + 3 = 6 - a + b$$

or $\boxed{10a - 4b = 6 - 3 = 3}$

which we should solve for a and b.

$$4a - 2b = 1$$
$$10a - 4b = 3$$

Solve any way you like. For example,

$$b = \frac{1}{2}(4a-1)$$

from the 1st equation, so the 2nd equation becomes

$$10a - 4\cdot\frac{1}{2}(4a-1) = 3$$

or $10a - 8a + 2 = 3$

or $2a = 1$ or $a = 1/2$

then $b = \frac{1}{2}(4a-1) = \frac{1}{2}(2-1) = 1/2$

Now, <u>check</u>:

$$f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ \frac{1}{2}x^2 - \frac{1}{2}x + 3 & \text{if } 2 \le x < 3 \\ 2x \underbrace{- \frac{1}{2} + \frac{1}{2}}_{0} & \text{if } 3 \le x \end{cases}$$

at $x = 2$: $f(x) = \frac{1}{2}\cdot 4 - \frac{1}{2}\cdot 2 + 3$

$= 2 - 1 + 3 = 4$

which agrees with $x+2$ there

at $x = 3$: $f(x) = 2\cdot 3 = 6$

and on the left, $\frac{1}{2}\cdot 9 - \frac{1}{2}\cdot 3 + 3$

$= \frac{6}{2} + 3 = 6$

which again agrees.

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#52: at $x = 1$,

$$\ln(x) - (3 - 2x)$$

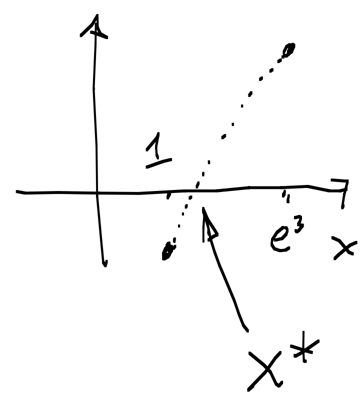
$$= 0 - (3-2) = -1$$

is negative;

at (say) $x = e^3$,

$$\ln(x) - (3 - 2x)$$

$$= \ln(e^3) - (3 - 2 \cdot e^3)$$



$$= 3 - (3 - 2 \cdot e^3)$$

$$= 3 - 3 + 2 \cdot e^3$$

$$= 2 \cdot e^3 \quad \text{is positive.}$$

$\therefore$ for some x^* in $1 \le x \le e^3$,

we have $\ln x^* = 3 - 2x^*$.

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BONUS

$$X^* = \frac{1}{2} W(2e^3)$$

$$\doteq 1.34996\ldots$$

as those of you who kept the Lambert W function poster might have been able to discover!

[$\ln x^* = 3 - 2x^*$ so $e^{\ln x^*} = e^{3-2x^*}$

or $x^* = e^{3-2x^*}$ or $x^* e^{2x^*} = e^3$

or $(2x^*) e^{(2x^*)} = 2e^3$

now $W(u) e^{W(u)} = u$ by definition

so $2x^* = W(2e^3)$ by pattern matching

or $x^* = \frac{1}{2} W(2e^3) \doteq 1.34996\ldots$]

$\ln x^* \doteq 0.3000763$ $\quad$ $3 - 2x^* \doteq 0.3000763$ 4